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Quaternionic Quantum Mechanics is Consistent with Complex Quantum Mechanics

C. G. Nash[1] and G. C. Joshi[1]

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Quaternionic quantum mechanics is investigated in the light of the great success of complex quantum mechanics. It is shown that to reproduce the results of complex quantum mechanics, quaternionic quantum mechanics must contain complex quantum mechanics.

1. INTRODUCTION

The work of Birkhoff and von Neumann (1936) was aimed at finding physically plausible hypotheses which imply that quantum theories must be described by a complex quantum mechanics. They were unable to find natural hypotheses which restrict quantum theory sufficiently. They were, however, able to find natural hypotheses which restrict quantum theories to those that are described by a lattice of propositions which is isomorphic to a lattice of closed subspaces of a Hilbert space over a division algebra.

Now complex quantum mechanics (CQM) is such a successful theory both in its accuracy of prediction and its scope of application that there have been many attempts to single it out by further restricting and better motivating the structure of possible quantum theories. However, the hypotheses of Birkhoff and von Neumann remain the most restrictive if not the most natural. We are therefore left to face the possibility of quantum theories different from and alternative to CQM. We require that any such alternative incorporate the success of CQM and explain its apparently universal scope. The accuracy of CQM predictions means that alternative theories must in essence extend CQM rather than differ from it. There appear to be two ways that an alternative quantum theory can extend CQM.

First, it may be that the alternative theory is only applicable to a restricted set of, as yet, unobserved particles either exotic or constituent,

[1]School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia.

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and that in some manner the effects of the alternative theory are confined to these particles so as to only weakly perturb the observed particles. This approach certainly avoids contradicting CQM, yet it does so at the cost of introducing some unspecified and undemonstrated mechanism. It is also unsatisfactory because of its *ad hoc* separation of CQM and the alternative theory. This separation does not treat the alternative theory as a true alternative in the spirit of Birkhoff and von Neumann, but rather as an addition to CQM which has been artificially prevented from contradicting CQM.

Second, it may be, and in the first place it seems preferable, that the alternative theory is taken as a true alternative and considered as applicable to all particles. If this is the case, we still need a mechanism whereby the alternative theory does not contradict CQM.

The fact that we are familiar with exotic behavior being restricted to areas that we have not yet observed does not lend any merit to the assumption made in the first approach instead of that made in the second. In the absence of a demonstration that either assumption is true, the second approach is to be preferred, as it fits in more with the Birkhoff and von Neumann idea of "alternative."

Since the work of Birkhoff and von Neumann, the lattice of subspaces of a quaternionic Hilbert space has been seen as an alternative realization of the lattice of propositions of quantum theory to the lattice of subspaces of a complex Hilbert space. Gleason's theorem and the spectral theorem carry over from complex quantum theory (Finkelstein *et al.*, 1959; Beltrometti and Cassinelli, 1981), and therefore in quaternionic quantum theory, just as in the complex theory, states may be identified with density operators and observables with Hermitian operators. So quaternionic quantum theory may be expressed in Hilbert space language as a direct alternative to the complex Hilbert space formulation of quantum theory. The second approach above is then to be preferred for the study of quaternionic quantum mechanics (QQM) in particular because it gives QQM the status of a true alternative theory, which it appears to be. In this paper we will take the second approach and we show that an immediate consequence of the requirement that QQM not contradict CQM is that QQM contains CQM, thus splitting QQM into complex and quaternionic parts. A mechanism by which these quaternionic parts do not contradict CQM is discussed by us elsewhere (Nash and Joshi, to appear).

2. CQM INSIDE QQM

Here we will show why and, to some extent, how the states and propositions of CQM correspond to states and propositions of QQM. That is to say that in a loose way, we find CQM inside QQM. This inclusion of CQM in QQM will be made stronger and more precise in Section 5.

We are now supposing that the world is completely described by QQM, that is to say that the preparation procedures (states) and the yes/no experiments (propositions) of the world correspond, respectively, to the density operators and the projection operators of a quaternionic Hilbert space. However, CQM has successfully associated at least a subset of these states and propositions with density and projection operators of a complex Hilbert space.

If we assume that every density operator and every projection operator that CQM would normally associate with a state or proposition of the world does indeed correspond to one in the world, then, as QQM also associates a density operator or a projection operator with that same state or proposition, there exist mappings h_p and h_s mapping projection operators of CQM to projection operators of QQM and mapping density operators of CQM to density operators of QQM, respectively.

If we are hoping that QQM avoids being in glaring and immediate conflict with the successes of CQM, then it is necessary that the above association exist and QQM agree with CQM when considering the states and propositions that CQM describes. This requires for a density operator D of CQM and a projection operator P_M mapping onto the subspace M of the complex Hilbert space that

$$T_r(DP_M) \simeq T_r(h_s(D)h_p(P_M)) \tag{2.1}$$

However, in this paper, we will assume equality for (2.1) above. This leaves the consideration of possible inequality in (2.1) for further papers, where we hope to show that slight inequality in (2.1) for any states and propositions leads to gross inequalities for others.

Position and momentum are such basic quantities that it is fair to demand their association with Hermitian operators in both CQM and QQM. Given the specific operators to which they correspond (Q_C, Q_Q, P_C, and P_Q), the propositions belonging to their spectral measures are identified. We will write these spectral measures as $P_{Q_C}(\cdot)$, $P_{Q_q}(\cdot)$, $P_{P_C}(\cdot)$, and $P_{P_Q}(\cdot)$, respectively. For the identified propositions of CQM to be consistent with the identified propositions of QQM, it must be for all Borel subsets E of R and for all states D of CQM that

$$\begin{aligned} \mathrm{Tr}(DP_{Q_C}(E)) &= \mathrm{Tr}(h_s(D)P_{Q_Q}(E)) \\ \mathrm{Tr}(DP_{P_C}(E)) &= \mathrm{Tr}(h_s(D)P_{P_Q}(E)) \end{aligned} \tag{2.2}$$

This then restricts h_S and h_P to satisfy

$$\begin{aligned} \mathrm{Tr}(h_S(D)h_P(P_{Q_C}(E))) &= \mathrm{Tr}(h_S(D)P_{Q_Q}(E)) \\ \mathrm{Tr}(h_S(D)h_P(P_{P_C}(E))) &= \mathrm{Tr}(h_S(D)P_{P_Q}(E)) \end{aligned} \tag{2.3}$$

The four equations above restrict the nature of h_S and h_P and, after examining the possible forms that position and momentum observables can take, form the basis of further discussion of h_S and h_P.

3. THE QQM OF A SINGLE PARTICLE IN ONE DIMENSION

The analysis in this paper relies on the work that has gone before, in that many useful results that are valid in CQM have been shown to be valid in QQM. In particular, Gleason's theorem and the spectral theorem apply to QQM (Finkelstein *et al.*, 1959; Beltrometti and Cassinelli, 1981; Horowitz and Biedenharn, 1984; Jauch, 1968; Truini *et al.*, 1981). The results already shown are sufficient to ensure, among other things, the efficacy of one-dimensional wave mechanics for an examination of the fundamentals of QQM. We will use it for such an examination.

So, following earlier work, the QQM of a particle in one dimension may be formulated in terms of the quaternionic Hilbert space $H_Q = L_Q^2(R)$, the space of Lebesque square-integrable functions from R to Q. The space and time translations are implemented by unitary operators $U_\alpha = \exp(\alpha\, \partial/\partial x)$ and $V_t = \exp(-JHt)$, respectively. H is a positive Hermitian operator and J is a skew-Hermitian unitary operator which commutes with H. J and H are uniquely determined by V_t and the positivity of H. The position and momentum operators are then x and $J'\,\partial/\partial x$, where J' is some, as yet undetermined, skew-Hermitian unitary operator which commutes with $\partial/\partial x$. Such freedom in the choice of momentum operator is not considered in the analogous CQM situation, where $J = J' = i$ is taken to be the case. A restriction on J and J' will, however, develop from the relationship between CQM and QQM.

The specific form of the spectral measure of position is as in the CQM situation, that is, for any Borel subset of R, E, any $u \in H_Q$, and all $x \in R$,

$$(P_{Q_Q}(E)u)(x) = \chi_E(x)u(x)$$

As J' is not restricted, we will not attempt to give the specific form of the spectral measure of momentum. We will, however, give the spectral measure of the operator $q\,\partial/\partial x$, where q is some unit pure imaginary quaternion. To this end, we define, for $u \in H_Q$,

$$\tilde{u}_q(p) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp(qpx)u(x)\, dx$$

Then for any Borel subset of R and any $u \in H_Q$

$$(P_{q\,\partial/\partial x}(E)u)(x) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-qpx)\chi_E(p)\tilde{u}_q(p)\, dp$$

The above expression may be associated with the spectral measure of P_C in the following way. We note that there exists a unit pure imaginary quaternion q_1 such that $[q_1, q]_+ = 0$. Then for any $u \in H_Q$, $u(x)$ may be written as $u^0(x) + u^1(x)q_1$ where $u^0, u^l \in L^2_{\langle\{1,q\}\rangle}(R) \cong H_C$ is the one-dimensional complex Hilbert space. Then

$$\begin{aligned}(P_{q\,\partial/\partial x}(E)u)(x) &= (P_{q\,\partial/\partial x}(E)u^0)(x) + (P_{q\,\partial/\partial x}(E)u^l)(x)q_1 \\ &= \sigma[(P_{P_C}(E)\sigma^{-1}(u^0))(x)] + \sigma[(P_{P_C}(E)\sigma^{-1}(u^l))(x)]q_1\end{aligned}$$

where σ represents both the isomorphism from C to $\langle 1, q\rangle$ and the associated isomorphism from H_C to $L^2_{\langle 1,q\rangle}(R)$.

4. CONVENTIONS

For an excellent review of the concepts of quaternionic Hilbert spaces we refer to Horwitz and Biedenharn (1984), whose conventions we will use. In particular, we note that we will define multiplication by scalars as on the right.

The following lemmas concern the relationship between one-dimensional wave CQM with one-dimensional wave QQM and when we refer to CQM and QQM it should be understood that it is to the one-dimensional wave mechanics. The distinction between the inner products of H_Q and H_C will not be explicitly made but should be taken from the context. The symbols $\langle\cdot\rangle$ will indicate the structure generated by the set they enclose, the nature of which, algebra or subspace, will be given by the context and the nature of the elements of the set.

We will use normalized vectors unless otherwise specified.

Projection operators mapping into one-dimensional subspaces will be denoted by P_v, where v is an element of the one-dimensional subspace.

For $v \in H_C$ and $u \in H_Q$ such that $v(x) \in RC$ and $u(x) \in RQ$ and $v(x) = u(x)$ for all $x \in R$ it will be convenient to denote both by the same symbol and write both as v.

5. LEMMAS

With the following lemmas we will show the requirement that QQM be consistent with CQM, in the sense discussed above, ensures that the entire structure of the Hilbert space formulation of CQM is found intact and with the same physical interpretation inside the Hilbert space formulation of QQM.

Lemma 1. For $v \in H_C$, that $\{p: \tilde{v}(p) \neq 0\}$ is bounded implies v is an element of the domain of $\partial^n/\partial x^n$ for $n = 1, 2, \ldots$, i.e., that $v \in C^\infty$.

For $u \in H_Q$, that $\{p: \tilde{u}_q(p) \neq 0\}$ is bounded for some pure imaginary unit quaternion q_1 implies u is an element of $\partial^n/\partial x^n$ for $n = 1, 2, \ldots$.

Proof. Take $v \in H_C$ such that $E = \{p: \tilde{v}(p) \neq 0\}$ is bounded. Therefore

$$v(x) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp(ipx)\, \tilde{v}(p)\, dp$$
$$= \left(\frac{1}{2\pi}\right)^{1/2} \int_{E} \exp(ipx)\, \tilde{v}(p)\, dp$$

Therefore

$$\frac{\partial^n}{\partial x^n} v(x) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{E} (ip)^n \exp(ipx)\, \tilde{v}(p)\, dp$$

The mapping $v \to \tilde{v}$ preserves the norm. Therefore as v is square-integrable, so is $\tilde{v}$. Now for v to be an element of the domain of $\partial^n/\partial x^n$ for all $n = 1, 2, \ldots, (\partial^n/\partial z^n)^v$ must be square-integrable for all those n. As $v \to \tilde{v}$ preserves the norm,

$$\int_{-\infty}^{\infty} \left| \frac{\partial^n}{\partial x} v(x) \right|^2 dx = \int_{-\infty}^{\infty} |(ip)^n \tilde{v}(p)|^2\, dp$$
$$\leq \int_E |\tilde{v}(p)|^2\, dp \cdot \sup_E(p^2)$$
$$< \infty \qquad \text{because } E \text{ is bounded}$$

So $(\partial^n/\partial x^n)^v$ is square-integrable for all $n = 1, 2, \ldots$.

For the second part of the lemma we take $u \in H_Q$ and q a pure imaginary unit quaternion such that $\{p: \tilde{u}_q(p) \neq 0\}$ is bounded. We can find another unit pure imaginary quaternion q_1 which anticommutes with q; therefore for all x we may express $u(x)$ as $u^0(x) + u^l(x) q_1$ with $u^0(x), u^l(x) \in \langle\{1, q\}\rangle$. Since u is square-integrable, then as

$$\infty > \int_{-\infty}^{\infty} |u(x)|^2\, dx = \int_{-\infty}^{\infty} |u^0(x)|^2\, dx + \int_{-\infty}^{\infty} |u^l(x)|^2\, dx$$

u^l and u^0 are also square-integrable. And therefore elements of $L^2_{\langle 1,q\rangle}(R)$ are isometrically isomorphic to H_C. Note that

$$\tilde{u}_q(p) = \left(\frac{1}{2\pi}\right)^{1/2} \int \exp(qpx)\, u(x)\, dx$$
$$= \int_{-\infty}^{\infty} \exp(qpx)\, u^0(x)\, dx + \int_{-\infty}^{\infty} \exp(qpx)\, u^1(x)\, dx q_1$$

and so by using σ as described in Section 3,

$$\tilde{u}_q(p) = (\sigma^{-1} u^0)(p) + (\sigma^{-1} u^l)(p) q_1$$

Then, as $\{p: \tilde{u}_q(p) \neq 0\}$ is bounded, so $\{p: (\sigma^{-1}u^0)(p) \neq 0\}$ and $\{p: (\sigma^{-1}u^1) \times (p) \neq 0\}$ are bounded. So u^0 and u^l are elements of the domain of $\partial^n/\partial x^n$ for $n = 1, 2, \ldots$, and therefore u is also. QED

Lemma 2. Take $v \in H_C$ and consider the state P_v of the CQM. Let $h_s(P_v) = \sum_i \alpha_i P_{u_i}$, where $u_i \in H_Q$ for all i. Then $E = \{p: \tilde{v}(p) \neq 0\}$ is bounded implies that for any unit pure imaginary quaternion q, $\{p: \tilde{u}_{iq}(p) \neq 0\}$ is bounded.

Proof. Take $v \in H_C$ such that $E = \{p: \tilde{v}(p) \neq 0\}$ is bounded. Then

$$\begin{aligned} 1 &= \mathrm{Tr}(P_v P_{P_C}(E)) \\ &= \mathrm{Tr}(h_S(P_v) P_{P_Q}(E)) \\ &= \sum_i \alpha_i \, \mathrm{Tr}(P_{u_i} P_{P_Q}(E)) \end{aligned}$$

Therefore, as $\sum_i \alpha_i = 1$ and all the α_i are positive,

$$\mathrm{Tr}(P_{u_i} P_{P_Q}(E)) = 1 \qquad \text{for all} \quad i \tag{5.1}$$

As $P_Q = J' \, \partial/\partial x$ with J' not necessarily equal to a fixed quaternion unit, (5.1) does not restrict u_i in a simple fashion. However, we may use (5.1) to restrict $q \, \partial/\partial x$. Note that for any self-adjoint operator A in either H_Q or H_C the relationship between the spectral measure of A and that of A^2 is $P_{A^2}(B) = P_A(B \cup -B)$ for any Borel subset of R, B. So for all i,

$$1 \geq \mathrm{Tr}(P_{u_i} P_{P_Q^2}(E)) = \mathrm{Tr}(P_{u_i} P_{P_Q}(E \cup -E)) \geq \mathrm{Tr}(P_{u_i} P_{P_Q}(E)) = 1$$

Therefore

$$\mathrm{Tr}(P_{u_i} P_{P_Q^2}(E)) = 1 \qquad \text{for all} \quad i$$

Now $P_Q^2 = -\partial^2/\partial x^2 = (q \, \partial/\partial x)^2$ for any unit pure imaginary quaternion q. Therefore for all i

$$\mathrm{Tr}(P_{u_i} P_{q \, \partial/\partial x}(E \cup -E)) = \mathrm{Tr}(P_{u_i} P_{P_Q^2}(E)) = 1$$

Therefore for all i, $\{p: \tilde{u}_{iq}(p) \neq 0\} E \cup -E$, is bounded because E is bounded. QED

Lemma 3. If for $v \in H_c$, $v(x)$ is real for all x, $\{p: \tilde{v}(p) \neq 0\}$ is bounded and $\{x: v(x) = 0\}$ is discrete, then for the state P_v in the H_C, $h_s(P_v) = P_v$ in H_Q.

Proof. Let v satisfy the conditions of the lemma and let $h_s(P_v) = \sum_i \alpha_i P_{u_i}$. Then for any Borel subset of R, E,

$$\mathrm{Tr}(P_v P_{Q_C}(E)) = \mathrm{Tr} \sum_i \alpha_i P_{u_i} P_{Q_Q}(E)) \tag{5.2}$$

Therefore $|v(x)|^2 = \sum_i \alpha_i |u_i(x)|^2$ for all x.

Therefore, for all x, $v(x)=0$ implies that $u_i(x)=0$ for all i. This means that for all i, $u_i(x)$ may be written as $f_i(x)v(x)$ for some function f_i from R to Q. Therefore, for all x

$$1=\sum_i \alpha_i f_i^*(x)f_i(x) \tag{5.3}$$

where * represents quaternion as well as complex conjugation.

Differentiating (5.3) for future use, we have

$$0=\sum_i \alpha_i\left(\frac{\partial}{\partial x}(f_i^*(x))f_i(x)+f_i^*(x)\frac{\partial}{\partial x}(f_i(x))\right) \tag{5.4}$$

From Lemma 1, $v\in \mathrm{Dom}(\partial^n/\partial x^n)$ in H_C for $n=1,2,\ldots$, and from Lemmas 1 and 2m $u_i\in \mathrm{Dom}(\partial^n/\partial x^n)$ in H_Q for $n=1,2,\ldots$, and all i. So the expectation values of P_C^2 and P_Q^2 for the states P_v and $\sum_i \alpha_i P_{u_i}$, respectively, may be written as $(v, P_C^2 v)$ and $\sum_i \alpha_i(u_i, P^2 u_i)$. Therefore from (5.2)

$$\left(v, -\frac{\partial^2}{\partial x^2}v\right)=\sum_i \alpha_i\left(u_i, -\frac{\partial^2}{\partial x^2}u_i\right)$$

and therefore

$$\left(\frac{\partial}{\partial x}v, \frac{\partial}{\partial x}v\right)=\sum_i \alpha_i\left(\frac{\partial}{\partial x}u_i, \frac{\partial}{\partial x}u_i\right) \tag{5.5}$$

Writing derivatives as primer and inner products as integrals, we have from (5.5), that

$$\int_{-\infty}^{\infty}(v'(x))^*v'(x)\,dx=\sum_i \alpha_i\int_{-\infty}^{\infty}(f_i'(x)v(x)+f_i(x)v'(x))^* \times(f_i'(x)v(x)+f_i(x)v'(x))\,dx$$

As $v(x)$ is real for all x, we then have that

$$\begin{aligned}\int_{-\infty}^{\infty}(v'(x))^2\,dx &= \int_{-\infty}^{\infty}\sum_i \alpha_i f_i'^*(x)f_i'(x)v(x)^2\,dx\\ &+\int_{-\infty}^{\infty}\sum_i \alpha_i f_i'^*(x)f_i(x)+f_i^*(x)f_i'(x))v'(x)v(x)\,dx\\ &+\int_{-\infty}^{\infty}\sum_i \alpha_i f_i^*(x)f_i(x)(v'(x))^2\,dx\end{aligned} \tag{5.6}$$

Therefore from (5.6)

$$0=\int_{-\infty}^{\infty}\sum_i \alpha_i f_i'^*(x)f_i'(x)v(x)^2\,dx$$

Therefore, as α_i is positive for all i, $f_i'(x)=0$ for all i and all x for which $v(x)\neq 0$.

If $f_i(x)$ is not constant, then, as $\{x: v(x)=0\}$ is discrete, it must be that $f_i(x)$ is discontinuous. This would imply that u_i is discontinuous because v is C^∞, but u_i is also C^∞, so for all i, $f_i(x)$ is a constant, say f_i. So $u_i(x)=v(x)f_i$ for all x. Therefore,

$$h_s(P_v)=\sum \alpha_i P_{vf_i}=\sum_i \alpha_i P_v = P_v$$

as $\sum \alpha_i = 1$. QED

Lemma 4. The set of real vectors $v\in H_C$ such that $h_s(P_v)=P_v$ contains a complete orthonormal set in H_C and so spans H_C.

Proof. Consider the set

$$B=\left\{v\in H_C: v(x)=\left(\frac{a}{\pi}\right)^{1/2}\frac{\sin(ax)}{ax} \quad \text{or} \quad v(x)=\left(\frac{a}{\pi}\right)^{1/2}\frac{1-\cos(ax)}{ax}\right.$$
$$\left.\text{where } a \text{ is any positive rational real}\right\}$$

If

$$v(x)=\left(\frac{a}{\pi}\right)^{1/2}\frac{\sin(ax)}{ax}$$

then $\tilde{v}(p)=(2a)^{-1/2}$ for $-a<p<a$, and $\tilde{v}(p)=0$ otherwise.

If

$$v(x)=\left(\frac{a}{\pi}\right)^{1/2}\frac{1-\cos(ax)}{ax}$$

then $\tilde{v}(p)=-i(2a)^{-1/2}$ for $-a<p<0$ and $\tilde{v}(p)=i(2a)^{-1/2}$ for $-a<p<0$, and $\tilde{v}(p)=0$ otherwise.

Now B is a linearly independent set of functions in H_C each of which satisfies the conditions of Lemma 3, that is, they are real, $\{p: \tilde{v}(p)\neq 0\}$ is bounded, and $\{x: v(x)=0\}$ is discrete. B spans H_C as the Fourier transforms span the set of all step functions.

From B we may produce a complete orthonormal set O via the Gram-Schmidt process. At each stage of this process use is made only of finite sums and scalar multiplication by inner products of the elements of B. These inner products are always real because all the elements of B are real. So each $o\in O$ satisfies the conditions of Lemma 3 and therefore $h_s(P_o)=P_o$ for all $o\in O$. QED

Lemma 5. For any state P_v of CQM, if $h_s(P_v)=P_{v'}$ for some $v'\in H_Q$, then for the corresponding proposition P_v of CQM $h_p(P_v)=P_{v'}$.

Proof. Take P_v a state of CQM such that $h_s(P_v) = P_{v'}$. Suppose h_p sends the proposition P_v to the proposition of QQM P_M for M a closed subspace of H_Q. Now

$$1 = \mathrm{Tr}(P_v P_v) = \mathrm{Tr}(h_s(P_v) h_p(P_v)) = \mathrm{Tr}(P_{v'} P_M) = (v', P_M v')$$

Therefore $v' \in M$. We can always find an orthonormal basis of M containing v'; let $\{v', m_1, m_2, \ldots\}$ be such a basis. Let O be the complete orthonormal basis of H_C produced in Lemma 4. Note that O may also be considered as a complete orthonormal basis of H_Q because it contains only real vectors which may be considered as elements of both spaces. Because $h_s(P_{o_i}) = P_{o_i}$ for every $o_i \in O$ we have for all i that

$$\begin{aligned} |(o_i, v)|^2 &= \mathrm{Tr}(P_{o_i} P_v) = \mathrm{Tr}(h_s(P_{o_i}) h_p(P_v)) = \mathrm{Tr}(P_{o_i} P_M) \\ &= (o_i, P_M o_i) = |(o_i, v')|^2 + \sum_j |(o_i, m_j)|^2 \end{aligned} \tag{5.7}$$

For the case that $v = v'$ we immediately have that $(o_i, m_j) = 0$ for all i and j. This implies that $M = \langle v \rangle$, i.e., that $h_p(P_v) = P_v$, because 0 spans H_Q.

For the case that $v \neq v'$ we note that the proof so far demonstrates that $h_p(P_{o_i}) = P_{o_i}$ for all i because $h_s(P_{o_i}) = P_{o_i}$. Therefore for all i

$$\begin{aligned} |(v, o_i)|^2 &= \mathrm{Tr}(P_v P_{o_i}) = \mathrm{Tr}(h_s(P_v) h_p(P_{o_i})) \\ &= \mathrm{Tr}(P_{v'} P_{o_i}) = |(v', o_i)|^2 \end{aligned}$$

Then from (5.7), $(o_i, m_j) = 0$ for all i and j and so $m = \langle v' \rangle$ and

$$h_p(P_v) = P_{v'} \quad \text{QED}$$

Lemma 6. For $v \in H_C$ if $v(x)$ is real for all x, then $h_s(P_v) = P_v$.

Proof. Take v a real function in H_C. From Lemma 4 we have a real orthonormal basis O of H_C that can be equally considered as one of H_Q. For which $o_i \in O$ implies that $h_s(P_{o_i}) = P_{o_i}$. We therefore can write v as $\sum_j o_i a_i$ with a_i real for all i. Now the finite sums $v_n = \sum_{i=1}^n o_i a_i$ satisfy the conditions of Lemma 3. So $h_s(P_{v_n}) = P_{v_n}$. Then from Lemma 5, $h_p(P_{v_n}) = P_{v_n}$. Suppose that $h_s(P_v) = \sum_i \alpha_i P_{u_i}$; then

$$\begin{aligned} |(v, v_n)|^2 &= \mathrm{Tr}(P_v P_{v_n}) = \mathrm{Tr}(h_s(P_v) h_p(P_{v_n})) \\ &= \sum_j \alpha_i \, \mathrm{Tr}(P_{u_i} P_{v_n}) = \sum_j \alpha_i |(u_i, v_n)|^2 \end{aligned}$$

As n goes to infinity, v_n goes to v and $|(v, v_n)|^2$ goes to 1, and so $|(u_i, v_n)|^2$ goes to 1 for all i because all the α_i are positive and $\sum_i \alpha_i = 1$. Therefore, $|(u_i, v)|^2 = 1$ for all i. Therefore, for all i there exists some $s_i \in Q$ such that $u_i = v s_i$. Therefore

$$h_s(P_v) = \sum_i \alpha_i P_v = P_v \quad \text{QED}$$

Lemma 7. For any $v \in H_C$, if v can be written as $v_0 + iv_1$ with v_0 and v_1 independent but not necessarily normalized real elements of H_C, then there is some unit quaternion s and some pure imaginary unit quaternion q for which

$$h_s(P_v) = P_{s(v_0+v_1q)}$$

Proof. Take v, v_0, $v_1 \in H_C$ as above and suppose that $h_s(P_v) = \sum_i \alpha_i P_{u_i}$. First, we will show that for all i, u_i is an element of the space spanned by v_1 and v_0 in H_Q, $\langle\{v_0, v_1\}\rangle_{H_Q} = \{v_0 a + v_1 b\colon a, b \in Q\}$. To do this, take any w a real element of H_C for which $(v, w) = 0$; then using Lemma 6, we have

$$\begin{aligned} 0 = |(v, w)|^2 = \operatorname{Tr}(P_v P_w) &= \operatorname{Tr}(h_s(P_v) h_p(P_w)) \\ &= \sum_i \operatorname{Tr}(P_{u_i} P_w) = \sum_i \alpha_i |(u_i w)|^2 \end{aligned}$$

Because α_i is positive for all i, $(u_i, w) = 0$ for all i. Therefore $(v, w) = 0$ if and only if $0 = (v_0, w) = (v_1, w)$. Then as the real elements of H_C and H_Q are in one-to-one correspondence via the obvious identification, we have for all i that $0 = (u_i, w)$ for all real w an element of H_Q for which $(v_0, w) = (v_1, w) = 0$. Call this set of w's the set $W \subseteq H_Q$. Now W contains a complete orthonormal set spanning $\langle v_0, v_1\rangle_{H_Q}^{\perp}$ because for an element of H_Q to be perpendicular to both v_0 and v_1, so must its real components. These real components are therefore elements of W. Therefore, $W^{\perp} = \langle\{v_0, v_1\}\rangle_{H_Q}^{\perp\perp} = \langle\{v_0, v_1\}\rangle_{H_Q}$ and we also know that all $u_i \in W$.

Second, we show that there is some u an element of $\langle\{v_0, v_1\}\rangle_{H_Q}$ such that for all i, $P_{u_i} = P_u$. To do this, consider P_v as a proposition of CQM and suppose $h_p(P_v) = P_M$ for M a subspace of H_Q. So

$$1 = \operatorname{Tr}(P_v P_v) = \operatorname{Tr}(h_s(P_v) h_p(P_v) = \sum_i \alpha_i \operatorname{Tr}(P_{u_i} P_M)$$

Therefore, as all the α_i are positive and $\sum_i \alpha_i = 1$, $u_i \in M$ for all i.

Now,

$$\begin{aligned} |(v_0, v)|^2 = \operatorname{Tr}(P_{v_0} P_v) &= \operatorname{Tr}(h_s(P_{v_0}) h_p(P_v)) \\ &= \operatorname{Tr}(P_{v_0} P_M) = |(v_0, P_M v_0)|^2 \end{aligned}$$

and similarly $|(v_1, v)|^2 = |(v_1, P_M v_1)|^2$. As $|(v_0, v)|^2$ and $|(v_1, v)|^2$ are neither both one nor both zero, $M\langle\{v_0 v_1\}\rangle_{H_Q}$ must be one-dimensional. Call it $\langle\{u\}\rangle_{H_Q}$, say. Then because u_i must be an element of $M\langle\{v_0 v_1\}\rangle_{H_Q} = \langle u\rangle_{H_Q}$, then $P_{u_i} = P_u$ for all i.

So $h_s(P_v) = P_u$ and, from Lemma 5, $h_p(P_v) = P_u$.

We take u to be normalized and write $u = v_0 a + v_1 b$ for some $a, b \in Q$. Now by considering $|(v_0, u)|^2$ and $|(v_1, u)|^2$, we have

$$\begin{aligned} |v_0|^2 + |(v_0, v_1)|^2 = |(v_0, v)|^2 &= |(v_0, u)|^2 \\ &= |v_0|^2 |a|^2 + |(v_0, v_1)|^2 |b|^2 \end{aligned}$$

and

$$|v_1|^2+|(v_0, v_1)|^2=|(v_1, v)|^2=|(v_1, u)|^2$$
$$=|v_1|^2|b|^2+|(v_0, v_1)|^2|a|^2$$

from which one can deduce with some manipulation that $1=|a|^2=|b|^2$.

Taking $1=(u, u)=(v, v)$, we find from an alternative expansion of $|(v_0, u)|^2$ that $0=2\operatorname{Re}(i)(v_0, v_1)=2\operatorname{Re}(a^*b)(v_0, v_1)$. Therefore, $a^*b=q$ for some unit pure imaginary quaternion q and so

$$u=v_0a+v_1b=a(v_0+v_1q) \quad \text{QED}$$

Lemma 8. There exists a fixed unit quaternion s and a fixed unit pure imaginary quaternion q such that for all v that can be written as v_0+v_1i, as in Lemma 7, $h_s(P_v)=P_{s(v_0+v_1q)}$.

Proof. Take $v, w\in H_C$ with $v=v_0+v_1i$ and $w=w_0+w_1i$ both satisfying the conditions of Lemma 7. Using Lemma 7, there exist unit quaternions s and t and unit pure imaginary quaternions q and p such that

$$h_s(P_v)=P_{s(v_0+v_1q)} \qquad \text{and} \qquad h_s(P_w)=P_{t(w_0+w_1p)}$$

Now

$$|(v, w)|^2=|(v_0, w_0)+(v_1, w_1)+(v_0, w_1)i-i(v_1, w_0)|^2$$
$$=|(v_0, w_0)|^2+|(v_1, w_1)|^2+|(v_0, w_1)|^2+|(v_1, w_0)|^2$$
$$+2[(v_0, w_0)(v_1, w_1)-(v_0, w_1)(v_1, w_0)]$$

and

$$|(v, w)|^2=\operatorname{Tr}(P_vP_w)=\operatorname{Tr}(h_s(P_v)h_p(P_w))$$
$$=|s(v_0+v_1q), t(w_0^2+w_1p)|$$

By a long expansion we find that this is equal to

$$(v_0, w_0)^2+(v_1, w_1)^2+(v_0, w_1)^2+(v_1, w_0)^2$$
$$+2[(v_0, w_0)(v_1, w_1)-(v_0, w_1)-(v_0, w_1)(v_1, w_0)]\operatorname{Re}(q_1^*p)$$

where $q_1=t^*sqs^*t$.

Combining the two expressions for $|(v, w)|^2$ above, we can deduce that either $q_1=p$ or

$$D(v, w)=(v_0, w_0)(v_1, w_1)-(v_0, w_1)(v_1, w_0)=0$$

In the case that $D(v, w)\neq 0$ and therefore $q_1=p$ we have that

$$t(w_0+w_1p)=w_0t+w_1tt^*sqs^*t=s(w_0+w_1q)s^*t$$

and so $h_s(P_w) = P_{s(w_0+w_1q)}$. In the case that $D(v, w) = 0$ and $\langle\{v_0, v_1\}\rangle_{H_Q}$ is perpendicular to $\langle\{w_0, w_1\}\rangle_{H_Q}$ we will show that a $w' \in H_C$ may be found satisfying the conditions of Lemma 7 such that $D(v, w') \neq 0$ and $D(w', w) \neq 0$. In this case $h_s(P_{w'}) = P_{s(w_0'+w_1'q)}$, which then implies that $h_s(P_w) = P_{s(w_0+w_1q)}$. To show this, take w_0' and w_1' both to be linear combinations of v_0, v_1, w_0, and w_1 such that neither of w_0' and w_1' is perpendicular to any of v_0, v_1, w_0, and w_1.

Choose w_0' and w_1' such that (w_1', v_0), (w_1', v_1), and (w_0', v_1) are fixed; then, because v_0 is not parallel to v_1, we can choose (w_0', v_0) so that $D(v, w') \neq 0$. Because $\langle\{v_0, v_1\}\rangle_{H_Q}$ is perpendicular to $\langle\{w_0, w_1\}\rangle_{H_Q}$, we can independently and similarly ensure that $D(w', w) \neq 0$ while keeping w_0 and w_1 nonparallel.

Now for any w and v we can find a $w'' \in H_C$ such $\langle\{w'', w_1\}\rangle_{H_Q}$ is perpendicular to both $\langle\{v_0, v_1\}\rangle_{H_Q}$ and $\langle\{w_0, w_1\}\rangle_{H_Q}$. So, using the above argument twice, we have that $h_s(P_w) = P_{s(w_0+w_1q)}$. QED

Vectors which do not satisfy the conditions of Lemma 7 may always be written as the product of a real vector and a complex number, $uc = u(c_0 + ic_1)$. Now $h_s(P_{uc}) = h_s(P_u) = P_u = p_{su\sigma(c)}$, where σ is an isomorphism between C and $\langle\{1, q\}\rangle_Q$ for any unit pure imaginary quaternion q and any unit quaternion s. Then, using Lemma 8, there exist some q and some s such that for all $v \in H_C$, $h_s(P_v) = P_{\phi(v)}$, where ϕ is the σ linear isometry mapping H_C into H_Q with $(\phi(v))(x) = s\sigma(v(x))$.

For mixed states D of CQM, with a decomposition into pure states of $\sum_i \alpha_i h_s P_{v_i}$, $h_s(D)$ must equal $\sum_i \alpha_i h_s(p_{v_i}) = \sum_i \alpha_i p_{\phi(v_i)}$ or else be in contradiction with equation (2.2). So ϕ has been shown to generate h_s entirely. The following lemma shows that ϕ also generates h_p entirely.

Lemma 9. For all propositions of CQM P_{M_C}, M_C a subspace of H_C, and all propositions of QQM P_{M_Q}, M_Q a subspace of H_Q, if for all pure states of CQM D,

$$\mathrm{Tr}(DP_{M_C}) = \mathrm{Tr}(h_s(D)P_{M_Q}) \tag{5.8}$$

then $M_Q = \langle\phi(M_C)\rangle_{M_Q}$ and as a direct consequence

$$h_p(P_{M_C}) = P_{\langle\phi(M_C)\rangle_{H_Q}}$$

Proof. Take P_{M_C} and P_{M_Q} propositions of CQM and QQM, respectively, satisfying (5.8).

Now for all $v \in H_C$,

$$\begin{aligned}(v, P_{M_C}v) &= \mathrm{Tr}(P_v P_{M_C}) = \mathrm{Tr}(h_s(P_v)P_{M_Q}) = \mathrm{Tr}(P_{\phi(v)}P_{M_Q})\\ &= (\phi(v), P_{M_Q}\phi(v))\end{aligned}$$

If $\lambda \in M_C$, then $(v, P_{M_C}v) = \|u\|^2 = (\phi(v), P_{M_Q}\phi(v))$. As ϕ is an isometry, $\phi(v) \in M_Q$. Therefore, as M_Q is closed, $\langle\phi(M_C)\rangle_{H_Q} \subseteq M_Q$.

Let ψ be a nonzero element of M_Q such that $\Psi \in \langle\phi(M_C)\rangle_{H_Q}$; then, as $\langle\phi(M_C)\rangle_{H_Q} \quad M_Q$, $\Psi' = \Psi - P_{\langle\phi(M_C)\rangle_{H_Q}}\Psi$ is an element of M_Q such that it is perpendicular to $\langle\phi(M_C)\rangle_{H_Q}$. We know $\phi(H_C)$ spans H_Q, so there exists a $v \in H_C$ such that $(\phi(v), \Psi') \neq 0$. Then $v' = v - P_{M_C}v$ is an element of H_C perpendicular to M_C such that $(\phi(v'), (\Psi') \neq 0$. Therefore, $(v', P_{M_C}, v') = 0$ and $(\phi(v'), P_{M_Q}, \phi(v')) \neq 0$. This is a contradiction, so it must be that $M_Q = \langle\phi(M_C)\rangle_{H_Q}$. QED

For A_C and A_Q Hermitian operators of H_C and H_Q, respectively, to represent the same physical observable in CQM and QQM, respectively, it must be that their spectral measures $P_{A_C}(\cdot)$ and $P_{A_Q}(\cdot)$ satisfy

$$\mathrm{Tr}(DP_{A_C}(E)) = \mathrm{Tr}(h_s(D)P_{A_Q}(E)) \tag{5.9}$$

for all states of CQM D and all Borel subsets of R. Therefore from Lemma 9 if $P_{A_C}(E) = P_{M_C}$ for some subspace M_C of H_C, then $P_{A_Q}(E) = P_{\langle\phi(M_C)\rangle_{H_Q}}$. Then from the uniqueness of the spectral resolution of A_Q it can be shown that $A_Q\phi(v) = \phi(A_C v)$ for all $v \in H_C$ in the domain of A_C.

This implies that the momentum is $P_Q = s\sigma(i)s^{-1}\,\partial/\partial\alpha$ and that $J = s\sigma(i)s^{-1}$.

Finkelstein *et al.* (1959) established that the propositions of QQM that commute with a fixed anti-Hermitian unitary operator are isomorphic with the proposition of CQM. We have essentially shown the reverse: that the set of propositions of QQM which have the same physical content as the set of propositions of CQM must all commute with a fixed anti-Hermitian unitary operator J and that J is as above.

The following lemma extends this type of situation to unitary operators of H_C and H_Q.

Lemma 10. Lemma 9 showed that in a particular case the operators of QQM are closely related to the operators of CQM. In the present lemma we expand this to include unitary operators U_Q of QQM and U_C of CQM which satisfy

$$|(U_Q\phi(v), \phi(u))| = |(U_C v, u)| \qquad \text{for all} \quad v, u \in H_C \tag{5.10}$$

If one intends to represent a symmetry of the world by U_Q in QQM and by U_C in CQM, then one can expect that the transformation probabilities associated with the symmetry in QQM are equal to the corresponding probabilities in CQM, which is what (5.10) asserts. The statement of Lemma 10 is:

If U_C is a unitary operator in the H_C and U_Q is a unitary operator in H_Q, and U_C together with U_Q satisfy (5.10), then there is some $k \in C$ such that $U_Q\phi(v) = \phi(kU_C v)$ for all $v \in H_C$.

Proof. Take $v \in H_C$ and let $u = U_C v$; then from (5.10)

$$|(U_Q\phi(v), \phi(U_C v))| = |(U_C v, U_C v)| = |(v, v)| = \|v\|^2$$

So $U_Q\phi(v) = \phi(U_C v)q_v$ for some unit quaternion q_v. Now take nonzero $v_1, v_2 \in H_C$ such that $(v_1, v_2) = 0$; then

$$U_Q\phi(v_1 + v_2) = \phi U_C(v_1 + v_2)q_{v_1+v_2} = \phi(U_C v_1)q_{v_1+v_2} + \phi(U_C v_2)q_{v_1+v_2}$$

and

$$U_Q\phi(v_1 + v_2) = U_Q\phi(v_1) + U_Q\phi(v_2) = \phi(U_C v_1)q_{v_1} + \phi(U_C v_2)q_{v_2}$$

As $(v_1, v_2) = 0$, we have that $(\phi(U_C, v_1), \phi(U_C, v_2)) = 0$. So, from above, $q_{v_1+v_2} = q_{v_1} = q_{v_2}$.

Now take nonzero $v \in H_C$ and nonzero $a \in C$, then

$$\phi(U_C v)\sigma(a)q_{va} = \phi(U_C va)q_{va} = U_Q\phi(va) = U_Q\phi(v)\sigma(a)$$
$$= \phi(U_C v)q_v\sigma(a)$$

So $q_{va} = \sigma(a)^{-1}q_v\sigma(a)$.

Now take nonzero $v_1, v_2 \in H_C$ with $(v_1, v_2) = 0$ and nonzero $b \in C$; then by the above, $q_{v_1} = q_{v_2}$ as $(v_1, v_2) = 0$ and $q_{v_1} = q_{v_2 b}$ as $(v_1, v_2 b) = 0$ and $q_{v_2 b} = \sigma^{-1}(b)q_{v_2}\sigma(b)$. Therefore, $q_{v_1} = \sigma(b)^{-1}q_{v_1}\sigma(b)$. So $[q_{v_1}, \sigma(b)] = 0$ for all nonzero $v_1 \in H_C$ and nonzero $b \in C$. As σ is an isomorphism from C onto some subalgebra of Q, $q_{v_1} \in \sigma(C)$ for all $v_1 \in H_C$. This then means that $q_{va} = q_v$ for all nonzero $v \in H_C$ and nonzero $a \in C$.

As H_C has a complete orthonormal basis, then as every element of the basis is perpendicular to every other, their q's are all the same. Therefore, for every nonzero $v_1, v_2 \in H_C$, $q_{v_1} = q_{v_2} = q \in \sigma(C)$. We may set $q_0 = q$ and so, writing $q = \sigma(k)$ for some $k \in C$,

$$U_Q(\phi v) = \phi(U_C v)\sigma(k) = \phi((U_C v)k) = \phi(kU_C v) \quad \text{QED}$$

This lemma is particularly helpful when investigating the relationship between the time translation operators of QQM and CQM, U_{t_Q} and U_{t_C}, respectively. The condition (5.10) can then be read as: the probability that a proposition P_v of CQM is found to be true when the system has evolved a time t from when it was in the state P_v is equal to the probability that the proposition $h_p(P_u)$ in QQM is found to be true when the system has evolved for a time t from when it was in the state $h_s(P_u)$. This is precisely how we require the QQM description to relate to the CQM description. Lemma 10 then implies that for all t there exists a $k_t \in C$ such that for all $v \in H_C$

$$U_Q\phi(v) = \phi(k_t U_{t_C} v) \qquad (5.11)$$

As we said in Section 3, Finkelstein *et al.* and Truini have shown that U_{t_Q} may be written as $\exp(JH_Q t)$ with J a unitary skew-Hermitian operator and H_Q a positive Hermitian operator both uniquely determined from U_{t_Q} by the requirement that H_Q be positive.

We have from the corollary to Lemma 9 that for all $v \in H_C$ in the domain of H_C

$$H_Q \phi v = \phi H_C v \tag{5.12}$$

Then the uniqueness of J and H_Q described above together with equations (5.11) and (5.12) is sufficient to ensure that $k_t = 1$ for all t and that $J = s\sigma(i)s^{-1}$.

6. CONCLUSION

The requirement that one-dimensional wave QQM give results consistent with one-dimensional wave CQM in the region that CQM describes is sufficient to ensure, using only measurable quantities, the existence of a σ-linear isometry ϕ from H_C into H_Q which recreates the entire structure of CQM within QQM. We can more clearly characterize the complexity of an operator if we first unitarily transform QQM so that $s = 1$. Then ϕ identifies H_C with $L^2_{\langle\{1, \sigma\gamma(i)\}\rangle}(\mathbb{R})$. So, as the quaternionic linearity of an operator implies its "complex" linearity on $\phi(H_C)$ and because $\phi(H_C)$ spans H_Q, an operator of QQM corresponds to an operator of CQM if and only if it leaves $\phi(H_C)$ invariant. This allows us to talk of complex states and propositions of QQM.

In Lemmas 5–9 the N-linear isometry ϕ is shown to generate h_s and h_p as a consequence of two assumptions only:

1. That $\mathrm{Tr}(DP_M) = \mathrm{Tr}(h_s(D) h_p(P_M))$.
2. That there exists a real basis of H_c such that $h_s(P_v) = P_v$ for all v in the basis.

Standard functional analysis theorems imbedding one Hilbert space into another were not directly applicable to this problem, as the first assumption above does not immediately translate into preservation of the inner product.

Lemmas 1–4 show the existence of such a basis. This is achieved by first finding a particular set of states of CQM (call this S) for which the position probability density function (PPDF) and the expectation value of the momentum squared (EVM2) for each state in S are sufficient to determine them uniquely. The requirement that QQM agree with CQM as described in Section 2 means that h_s must map these states to states of QQM that have the same PPDF and EVM2. The form of the position and momentum operators of QQM as determined in Section 3 is sufficiently close to their form in CQM and the PPDFs and EVM2s of the states of S are sufficiently peculiar that when the PPDFs and EVM2s are considered in QQM they uniquely determine states which are identical to those states they determined in CQM, i.e., for P_V in S, $h_S(P_V) = P_V$. It is then shown that S contains a basis of H_C and therefore of H_Q.

The unique relationship between position and momentum, which is a direct result of relating the generator of space translations to momentum, is the peculiar premise (in terms of standard functional analysis) which ensures the existence of the basis. However, because of this grounding of our results in translation invariance we do not foresee any difficulty in extending our results to three dimensions, where this relationship remains between the fundamental observables positions and momentum. On the other hand, we do anticipate problems with extending our results to include internal symmetries and the like, where there is no such relationship between the observables nor is there the same sense of their fundamental nature.

We have not yet shown that QQM does not contradict CQM in any way (Truini *et al.*, 1981), but we have specified the structure of QQM for there to be a possibility of it being consistent with CQM (which is the world around us).

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